RELATIVE CONVOLUTIONS. I
PROPERTIES AND APPLICATIONS

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Abstract. To study operator algebras with symmetries in a wide sense we introduce a notion of *relative convolution operators* induced by a Lie algebra. Relative convolutions recover many important classes of operators, which have been already studied (operators of multiplication, usual group convolutions, two-sided convolution etc.) and their different combinations. Basic properties of relative convolutions are given and a connection with usual convolutions is established.

Presented examples show that relative convolutions provide us with a base for systematical applications of harmonic analysis to PDO theory, complex and hypercomplex analysis, coherent states, wavelet transform and quantum theory.

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Contents

1. Introduction

Convolution operators and different operators associated with them are very important in mathematics (for example, in complex analysis [24]) and physics (for example, quantum mechanics [21] and [55, Chap. 3]). The fundamental role of such operators may be easily explained. Indeed, a notion of symmetries and group transforms invariance are the basis of the contemporary science. It is well known, that operators, which are invariant under a transitive group operation, may be realized as convolution operators on the group.

However, there are some limitations for an application of convolution operators:

- It is not a rare case when a operators symmetry group and functions domain have different dimensions.
- Group convolution operators are generated by transformations of function domains. However, mathematical objects are often connected not only with domain transforms but also with alterations of function range or even both of them.

The paper introduces a notion of *relative convolutions*, which allows us to overcome these limitations. Naturally, such an interesting object cannot be totally unknown in mathematics and we will give a short description of connected ones in Remark 2.2. In this paper we work only with continuous groups of symmetry. Discrete groups or their mixtures with continuous ones may be also considered.

A geometric group structure and related objects (especially the non-commutative Fourier transform) are obviously determining for properties of convolution algebra. If the group structure is used not only as a basis for particular calculations but also in a more general framework, then it is possible to find the proper level of generality for obtained results[1].

[1]See, for example, the deduction of a PDO–form for convolutions on step 2 nilpotent Lie groups in [51, p. 9–10] or Theorem 5.8.

Be found, that geometry of Lie groups and algebras jointly with properties of kernels determine representations of relative convolution algebras. Namely, all representations of relative convolution algebras will be induced by selected representations of corresponding Lie algebras (Lie groups) and the selection of representations will depend on kernels properties. The representation theory of Lie algebras is complicated and still unsolved completely. But, due to the author personal interest, the main examples are nilpotent Lie algebras fully described by the Kirillov theory (see [31] or [52, Chap. 6]). Considered examples will show that such a restriction still leaves enough space for very interesting applications.

The layout is as follows.

In Section 2 we give necessary notations, introduce the main object of the paper — *relative convolution* — and show basic examples. Relative convolutions recover many important classes of operators, which have been already studied (operators of multiplication, usual group convolutions, two-sided convolutions etc.) and their different combinations.

Basic properties of relative convolutions are given in Section 3. We prove a formula for the composition of relative convolutions and deduce from it that *an algebra of relative convolutions induced by a Lie algebra* $\mathfrak{g}$ *is a representation of the algebra of group convolutions on the Lie group* $\operatorname{Exp}\mathfrak{g}$. We also state the universal role PDO and the Heisenberg group in the theory of relative convolution operators.

Relative convolutions provide us with a tool for the systematic usage of harmonic analysis in different fields of pure and applied mathematics. This statement will be illustrated in Sections 4 and 6 by many examples, which show applications of relative convolutions to the theory of PDO, complex and hypercomplex analysis, coherent states, wavelets and quantum mechanics.

In Section 5 we will describe our view on a notion of coherent states. The main observation is: a direct employment of group structures of coherent states gives us the uniform theory for the Bargmann, Bergman and Szegö projectors at the Segal–Bargmann (Fock), Bergman and Hardy spaces respectively. Applications to wavelets (and other) theory are also possible.

The author gratefully acknowledge his inspiration by papers [17, 18, 28, 29, 51]. Besides, there is some interference with the coherent recent paper of Folland [22]. The author is also grateful to Dr. V. V. Kravchenko and Prof. N. L. Vasilevski for helpful discussions.

2. Relative Convolutions

2.1. **Definitions and Notations.** Let G be a connected simply connected Lie group, let $\mathfrak{g}$ be its finite-dimensional Lie algebra. By the usual way we can identify $\mathfrak{g}$ with $\mathbb{R}^N$ for $N = \dim \mathfrak{g}$ as vector spaces. The exponential map $\exp : \mathfrak{g} \to G$ [32, § 6.4] identifies the group G and its algebra $\mathfrak{g}$. So G as a real C^∞-manyfold has the dimension N. Let fix a frame $\{X_j\}_{1\le j\le N}$ of $\mathfrak{g}$. Via the exponential map we can write a decomposition $g = \sum_1^N x_j X_j$ in *exponential coordinates* for every $g \in G$. The group law on G in the exponential coordinates may be expressed via the *Campbell–Hausdorff formula* [32, § 6.4]:

$$g * h = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{\substack{k_j+l_j\ge 1\\ k_j\ge 1,\, l_j\ge 1}} \frac{[x^{k_1}y^{l_1}\dots x^{k_m}y^{l_m}]}{k_1!l_1!\dots k_m!l_m!}, \tag{2.1}$$

where $[x_1x_2\dots x_n] = \frac{1}{n}[\dots[[x_1,x_2],x_3],\dots,x_n]$ and

$$g = \sum_1^N x_j X_j,\ h = \sum_1^N y_j X_j.$$

It seems reasonable to introduce a short notation for the right side of (2.1), we select $\mathrm{CH}\left[\sum_1^N x_j X_j, \sum_1^N y_j X_j\right]$ or $\mathrm{CH}\,[x,y]$ (if the frame is obvious).

Let S be a set and let be defined an operation $G : S \to S$ of G on S. If we fix a point $s \in S$ then the set of elements $G_s = \{g \in G \mid g(s) = s\}$ obviously forms the *isotropy (sub)group of s in G* [39, § I.5]. There is an equivalence relation on S, say, $s_1 \sim s_2 \Leftrightarrow \exists g \in G : gs_1 = s_2$, with respect to which S is a disjoint union of distinct orbits [39, § I.5]. Thus from now on, without lost of a generality, we assume that the operation of G on S is *transitive*, i. e. for every $s \in S$ we have

$$Gs := \bigcup_{g\in G} g(s) = S.$$

If G is a Lie group then the *homogeneous space* G/G_s is also a Lie group for every $s \in S$. Therefore the one-to-one mapping $G/G_s \to S : g \mapsto g(s)$ induces a structure of C^∞-manifold on S. Thus the class $C_0^\infty(S)$ of smooth functions with compact supports on S has the evident definition.

A smooth measure $d\nu$ on S is called *invariant* (the *Haar measure*) with respect to an operation of G on S if

$$\int_S f(s)d\nu(s) = \int_S f(g(s))d\nu(s),\ \text{for all } g \in G,\ f(s) \in C_0^\infty(S).$$

The Haar measure always exists and is uniquely defined up to a scalar multiplier [52, § 0.2]. An equivalent formulation of (2.2) is: *G operate on $L_2(S, d\nu)$ by unitary operators*. We will transfer the Haar measure $d\mu$ from G to $\mathfrak{g}$ via the exponential map $\exp : \mathfrak{g} \to G$ and will call it as the *invariant measure on a Lie algebra* $\mathfrak{g}$.

Now we can define an operation of a Lie algebra $\mathfrak{g}$ in the space $C_0^\infty(S)$ induced by an operation of G on S. Let $X \in \mathfrak{g}$ and $f(s) \in C_0^\infty(S)$ then

$$\lim_{t\to 0} \frac{f(e^{tX}s) - f(s)}{it} \in C_0^\infty(S), \tag{2.2}$$

where $\exp : X \mapsto e^{tX}$ is the exponential map $\exp : \mathfrak{g} \to G$. The value of limit (2.2) will be denoted by $[Xf](s)$. If S is equipped by a measure $d\nu$ we can define an *adjoint* operation X^* of $\mathfrak{g}$ on $L_2(S, d\nu)$ by natural formula $\langle f(s), [X^*g](s)\rangle := \langle [Xf](s), g(s)\rangle$. The invariance (2.2) of $d\nu$ may be reformulated in the terms of the Lie algebra $\mathfrak{g}$ as follows:

$$X_j^* = X_j,$$

for every $X_j,\ 1 \le j \le N$ from a frame of $\mathfrak{g}$. Summarizing, $X : f \to [Xf]$ is a selfadjoint (possibly unbounded) operator in $L_2(S, d\nu)$ with invariant measure $d\nu$.

In general, we will speak on an *operation* of a Lie algebra $\mathfrak{g}$ on a manifold S with a measure $d\nu$ if there is a linear representation of $\mathfrak{g}$ by selfadjoint operators on the linear space $L_2(S, d\nu)$. As usual, if X is selfadjoint on L_2 then e^{itX} is unitary on L_2. Clearly, every operation of a Lie group on S induces an operation of the corresponding Lie algebra, but inverse is not true generally speaking (see Example 2.6). Therefore we will based on the notion of a *Lie algebra* operation.

The succeeding object will be useful in our study of convolution algebras representations. We define the *kernel* of operation of $\mathfrak{g}$ on S as follows:

$$\operatorname{Ker}(\mathfrak{g},S)=\{X\in\mathfrak{g}\mid [Xf](x)=0 \text{ for all } f(x)\in C_0^\infty(S)\}. \tag{2.3}$$

If an operation of $\mathfrak{g}$ on S is induced by an operation of G on S then $X\in\operatorname{Ker}(\mathfrak{g},S)$ if and only if for $e^X\in G$ and for any $s\in S$ we have $e^X(s)=s$. There is no doubt that $\operatorname{Ker}(\mathfrak{g},S)$ is a two-sided ideal of $\mathfrak{g}$ (and therefore a linear subspace). Thus we can introduce the quotient Lie algebra $\operatorname{Ess}(\mathfrak{g},S)=\mathfrak{g}/\operatorname{Ker}(\mathfrak{g},S)$ (see [32, § 6.2]). An induced action of $\operatorname{Ess}(\mathfrak{g},S)$ on S is evidently specified.

Now we can describe the main object of the paper.

Definition 2.1. Let we have a (selfadjoint) operation of a Lie algebra $\mathfrak{g}$ on S (possibly induced by an operation of a Lie group G on a set S) and let $\{X_j\}_{1\le j\le N}$ be a fixed frame of $\mathfrak{g}$. We will define the *operator of relative convolution* K induced by $\mathfrak{g}$ on $E=C_0^\infty(S)$ with a kernel $k(x)\in C_0^\infty(\mathbb{R}^N)$ by the formula

$$K=(2\pi)^{-N/2}\int_{\mathbb{R}^N}\widehat{k}(x_1,x_2,\ldots,x_N)\,e^{i\sum_1^N x_jX_j}\,dx, \tag{2.4}$$

where integration is made with respect to an invariant measure on $\mathfrak{g}\cong\mathbb{R}^N$. Here $\widehat{k}(s)$ is the Fourier transform of the function $k(s)$ over $\mathbb{R}^N$:

$$\widehat{k}(x)=(2\pi)^{-N/2}\int_{\mathbb{R}^N}k(y)\,e^{-iyx}\,dy.$$

Remark 2.2. This definition has an origin at the *Weyl functional calculus* [54] (or the Weyl quantization procedure), see Example 4.1 for details. Feynman in [19] proposed its extension — the *functional calculus of ordered operators* — in a very similar way. Anderson [1] introduced a generalization of the Weyl calculus for arbitrary set of self-adjoint operators in a Banach space exactly by formula (2.4). A description of different operator calculuses may be found in [43]. But it was shown by R. Howe [28, 29], that success of the original Weyl calculus is intimately connected with the structure of the Heisenberg group and its different representations. Thus one can obtain a new fruitful branch in this direction making an assumption, that the operators X_j in (2.4) are not arbitrary but are connected with some group structure. Such a treatment for the Heisenberg group and multipliers may be found at the Dynin paper [17]. M. E. Taylor in [51] introduced "smooth families of convolution operators", which technically coincide with relative convolutions in many important cases. However, full symmetry groups of such smooth families are not clear and thus cannot be exploited. Efforts to study simultaneously both the group action and operators of multiplication bring also a very similar object at the recent paper[2] of Folland [22]. But general consideration of relative convolutions seems to be new (as well as systematical applications to new and already solved problems, see Sections 4 and 6).

Due to paper [1] our definition is correct for any representation of a Lie algebra in a Banach space. Thus we can use similar

Definition 2.3. Let we have a selfadjoint representation of a Lie algebra $\mathfrak{g}$ on a Banach space B and let $\{X_j\}_{1\le j\le N}$ be operators represented a fixed frame of $\mathfrak{g}$.

[2]In the mentioned paper consideration is restricted to nilpotent step 3 Lie algebras.

We will define the *operator of relative convolution* K induced by $\mathfrak{g}$ on B by the formula

$$K = (2\pi)^{-N/2} \int_{\mathbb{R}^N} \widehat{k}(x_1, x_2, \ldots, x_N)\, e^{i\sum_1^N x_j X_j}\, dx. \tag{2.5}$$

Here integration is made with respect to an invariant measure on $\mathfrak{g} \cong \mathbb{R}^N$.

Remark 2.4. As was shown at [1], formula (2.5) may be treated as a definition of a function $k(X_1, X_2, \ldots, X_N)$ of operators X_j. See Remark 4.11 for an alternative Riesz-like definition of relative convolutions.

We have defined relative convolutions only for a very restricted class of kernels $k(x)$ and a specific space E. Of course, both of them may be enlarged in the proper context. Another interesting modification may be required by a consideration of discrete groups and their actions. It may be achieved by a replacing integration in (2.4) by summation or integration by discrete measures. But in this paper we will not consider such cases.

As we will see, relative convolutions are naturally defined not only for group operation on *functions domains* but also on *ranges of function* (see Examples 2.6, 4.4 and 4.7).

2.2. **Basic Examples.** The following Examples make clear the relationships between the relative convolutions and usual ones.

Example 2.5. Let $G = \mathbb{R}^N$ and G operates on $S = \mathbb{R}^N$ as a group of Euclidean shifts $g : y \to y + g$. The algebra $\mathfrak{g}$ consists of selfadjoint differential operators spanned on the frame $X_j^e = \frac{1}{i}\frac{\partial}{\partial y_j}$, $1 \le j \le N$. Then:

$$\begin{aligned}
[Kf](y) &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} \widehat{k}(x)\, e^{i\sum_1^N -x_j(\frac{1}{i}\frac{\partial}{\partial y_j})} f(y)\, dx \\
&= (2\pi)^{-N/2} \int_{\mathbb{R}^N} \widehat{k}(x)\, e^{-\sum_1^N x_j \frac{\partial}{\partial y_j}}\, f(y)\, dx \\
&= (2\pi)^{-N/2} \int_{\mathbb{R}^N} \widehat{k}(x)\, f(y - x)\, dx.
\end{aligned}$$

Otherwise, an operator of relative convolution with a kernel $k(x)$ evidently coincides with a usual (Euclidean) convolution on $\mathbb{R}^N$ with the kernel $\widehat{k}(x)$.

This Example can be obviously generalized for an arbitrary Lie group G and the set $S = G$ with the natural left[3] (or right) operation of G on $S = G$

$$G : S \to GS\ (SG) : g' \mapsto g^{-1}g'\ (g' \mapsto g'g),\ g \in G,\ g' \in S = G.$$

It is clear, that relative convolutions will coincide with the left (right) group convolutions on G (with, may be, transformed kernels).

Example 2.6. Let $G = \mathbb{R}^N$ with the Lie algebra spanned on operators of multiplication $X_j = M_{y_j}$ by y_j, $1 \le j \le N$ on a space of functions on $S = \mathbb{R}^N$. Then a relative convolution K with a kernel $k(x)$

$$\begin{aligned}
[Kf](y) &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} \widehat{k}(x)\, e^{ixy}\, f(y)\, dx \\
&= k(y)f(y)
\end{aligned}$$

[3]For a commutative group G the left and the right operations are the same.

is simply an operator of multiplication by $k(y)$. In this case the operation of the Lie algebra $\mathfrak{g}$ on S is not generated by an operation of a Lie group on S. Of course, it is possible to establish a connection with Example 2.5 through the Fourier transform, but it is not so simple in other cases. Particularly it will happen when one generalizes this Example by use more complicated transformations of range than a multiplication by scalars (see Example 4.7) or simultaneously applies transformations from this and the previous Examples (see Example 4.1).

Example 2.7. Let $\mathbb{H}^n$ be the Heisenberg group [17, 51, 52]. The Heisenberg group $\mathbb{H}^n$ is a step 2 nilpotent Lie group. As a $C^\infty-$manifold it coincides with $\mathbb{R}^{2n+1}$. If an element of it is given in the form $g = (u, v) \in \mathbb{H}^n$, where $u \in \mathbb{R}$ and $v = (v_1, \ldots, v_n) \in \mathbb{C}^n$, then the group law on $\mathbb{H}^n$ can be written as

$$(u,v) * (u',v') = \left(u + u' - \frac{1}{2}\mathrm{Im} \sum_1^n v'_k \bar{v}_k,\ v_1 + v'_1, \ldots, v_n + v'_n \right). \tag{2.6}$$

We single out on $\mathbb{H}^n$ the group of nonisotropic dilations $\{\delta_\tau\}$, $\tau \in \mathbb{R}_+$:

$$\delta_\tau(u,v) = (\tau^2 u, \tau v).$$

Functions with the property

$$(f \circ \delta_\tau)(g) = \tau^k f(g) \tag{2.7}$$

will be called *δ_τ-homogeneous functions of degree k.*

The left and right Haar measure on the Heisenberg group coincides with the Lebesgue measure. Let us introduce the right π_r and the left π_l regular representations of $\mathbb{H}^n$ on $L_2(\mathbb{H}^n)$:

$$[\pi_l(g)f](h) = f(g^{-1} * h), \tag{2.8}$$

$$[\pi_r(g)f](h) = f(h * g). \tag{2.9}$$

Thereafter, $G = \mathbb{H}^n \times \mathbb{H}^n$, $S = \mathbb{H}^n$ and an operation of G on S is defined by the *two-sided* shift

$$G : s \to g_1^{-1} * s * g_2,\ \ s \in S = \mathbb{H}^n, (g_1, g_2) \in G = \mathbb{H}^n \times \mathbb{H}^n.$$

The Lie algebra of G is the direct sum $\mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{h}_n$ of two copies of the Lie algebra $\mathfrak{h}_n$ of the Heisenberg group $\mathbb{H}^n$. If in $S \cong \mathbb{R}^N$ we define Cartesian coordinates y_j, $0 \le j < N = 2n+1$, then a frame of $\mathfrak{g}$ may be written as follows

$$X_0^l = \frac{1}{i}\frac{\partial}{\partial y_0}, \quad X_j^l = \frac{1}{i}\left(\frac{\partial}{\partial y_{j+n}} - 2y_j \frac{\partial}{\partial y_0}\right), \quad X_{j+n}^l = \frac{1}{i}\left(\frac{\partial}{\partial y_j} + 2y_{j+n} \frac{\partial}{\partial y_0}\right), \tag{2.10}$$

$$X_0^r = -\frac{1}{i}\frac{\partial}{\partial y_0}, \quad X_j^r = \frac{1}{i}\left(\frac{\partial}{\partial y_{j+n}} + 2y_j \frac{\partial}{\partial y_0}\right), \quad X_{j+n}^r = \frac{1}{i}\left(\frac{\partial}{\partial y_j} - 2y_{j+n} \frac{\partial}{\partial y_0}\right), \tag{2.11}$$

where $1 \le j \le n$. Vector fields X_j^l, $0 \le j < N$ generate the left shifts (2.8) on $S = \mathbb{H}^n$ and X_j^r, $0 \le j < N$ generate the right ones (2.9). X_j^l (correspondingly X_j^r) satisfy to the famous Heisenberg commutation relation

$$[X_j^{l(r)}, X_{j+n}^{l(r)}] = -[X_{j+n}^{l(r)}, X_j^{l(r)}] = X_0^{l(r)} \tag{2.12}$$

and all other commutators being zero. Particulary, all X_j^l commute with all X_j^r.

Vector fields X_0^l and $-X_0^r$ are different as elements of $\mathfrak{g}$, but have coinciding operations on functions. It is easy to see, that $\operatorname{Ker}(\mathfrak{g}, S)$ is a linear span of the vector $X_0^l + X_0^r$ and $\operatorname{Ess}(\mathfrak{g}, S) = \mathbb{H}^{2n}$. Now formula (2.4) with kernel $k(x),\ x \in \mathbb{R}^{4n+2}$ defines the *two-sided convolutions* studied in [53]:

$$\begin{aligned}
[Kf](y) &= (2\pi)^{-N} \int_{\mathbb{R}^{4n+2}} \widehat{k}(x)\, e^{i\sum_1^{2N} x_j X_j} f(y)\, dx \\
&\overset{(*)}{=} (2\pi)^{-N} \int_{\mathbb{R}^{4n+2}} \widehat{k}(x)\, e^{i\sum_0^{N-1} x'_j X_j^l}\, e^{i\sum_0^{N-1} x''_j X_j^r} f(y)\, dx \\
&= (2\pi)^{-N} \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \widehat{k}(x', x'')\, f(x'^{-1} * y * x'')\, dx'dx'' \\
&= (2\pi)^{-N} \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \widehat{k}(x', x'')\pi_l(x')\pi_r(x'')\, dx'dx''\, f(y).
\end{aligned}$$

Transformation $(*)$ is possible due the commutativity of X_j^l and X_j^r.

Reduction of two-sided convolutions to the usual group ones was done in [36] in a way very similar to the present consideration (see Corollary 3.5).

The Heisenberg group here may be substituted by any non-commutative group G (for a commutative group the left and the right shifts are the same) and we will obtain two-sided convolutions on G.

These basic Examples form a frame for other ones: the number of examples may be increased both by simple compositions of convolutions from Examples 2.5–2.7 and by alterations of considered groups.

3. Basic Properties

The main purpose of the present introduction to relative convolutions is a sharp extension of harmonic analysis applications. Through relative convolutions we can transfer the knowledge on Lie groups and their representations to the different operator algebras. Properties of relative convolutions themselves are strictly depending on a structure of a concrete group. Thus, it seems unlikely that one can say too much about them in general. Nevertheless, results collected in this Section establish important properties of relative convolutions and will be forceful in the future.

$\operatorname{Ker}(\mathfrak{g}, S)$ operates on S trivially therefore we are interested to understand the effective part of a relative convolution operator. Let $m := \dim \operatorname{Ker}(\mathfrak{g}, S) > 0$. We have a decomposition $\mathfrak{g} = \operatorname{Ker}(\mathfrak{g}, S) \oplus \operatorname{Ess}(\mathfrak{g}, S)$ as linear spaces. Let us introduce a frame $\{X_j\}_{1\le j\le N}$ of $\mathfrak{g}$ such that $X_j \in \operatorname{Ker}(\mathfrak{g}, S), 1 \le j \le m$ and we assume that operator (2.4) is written through such a frame (it clearly can be obtained by a change of variables). Then the following Lemma is evident.

Lemma 3.1. (Effective decomposition) *Operator* (2.4) *of relative convolution is equal to an operator of relative convolution generated by the induced operation of* $\operatorname{Ess}(\mathfrak{g}, S)$ *on* S *with the kernel* $k'(x'')$ *defined by the formula*

$$\widehat{k'}(x'') = (2\pi)^{-m/2} \int_{\mathbb{R}^m} \widehat{k}(x', x'')\, dx', \tag{3.1}$$

where $x' \in \mathbb{R}^m,\ x'' \in \mathbb{R}^{N-m}$.

The effective decomposition allows us to consider always a basic case of $\operatorname{Ker}(\mathfrak{g}, S) = 0,\ \operatorname{Ess}(\mathfrak{g}, S) = \mathfrak{g}$. After that the general case may be obtained by the obvious modification.

Operators of relative convolution clearly form an algebra and we will denote by $(k_2 * k_1)(x)$ the kernel of composed operator of two relative convolutions with kernels $k_1(x)$ and $k_2(x)$. Of course, $k_2 * k_1 \neq k_1 * k_2$ generally speaking.

Proposition 3.2. (Composition formula) *We have*

$$(k_2 * k_1)(x) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} k_2(y)\, k_1(\mathrm{CH}\,[-y, x])\, dy, \tag{3.2}$$

where $\mathrm{CH}\,[-y, x] = \mathrm{CH}\,[y^{-1}, x]$ *is given by* (2.1).

Proof. For kernels k_1, k_2 satisfying the Fubini theorem we can change theintegration order and obtain formula (3.2):

$$\begin{aligned} K_2 K_1 &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} k_2(y)\, e^{i\sum_1^N y_j X_j}\, dy \times (2\pi)^{-N/2} \int_{\mathbb{R}^N} k_1(z)\, e^{i\sum_1^N z_j X_j}\, dz \\ &= (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} k_2(y)\, k_1(z)\, e^{i\sum_1^N y_j X_j} e^{i\sum_1^N z_j X_j}\, dy\, dz \\ &= (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} k_2(y)\, k_1(z)\, e^{i\mathrm{CH}\,[\sum_1^N y_j X_j, \sum_1^N z_j X_j]}\, dy\, dz \\ &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} (2\pi)^{-N/2} \int_{\mathbb{R}^N} k_2(y)\, k_1(\mathrm{CH}\,[-\sum_1^N y_j X_j, \sum_1^N x_j X_j])\, dy \\ &\qquad \times e^{i\sum_1^N x_j X_j}\, dx, \end{aligned}$$

where $e^{i\sum_1^N x_j X_j} = e^{i\sum_1^N y_j X_j} e^{i\sum_1^N z_j X_j}$. Making the change of variables we used the invariance of the measure on $\mathfrak{g}$. □

Note, that for an operation $\mathfrak{g}$ on $S = G$ induced by the operation of G on S formula (3.2) gives us the usual *group convolution* on $S = G$. Thus we can speak about a *convolution-like* calculus of relative convolutions. This remark gives us the following important result describing the nature of relative convolution algebras[4].

Theorem 3.3. *Let* $\mathfrak{g}$ *be an algebra Lie operating on a set* S *and let* G *be the exponential Lie group of* $\mathrm{Ess}\,(\mathfrak{g}, S)$. *Let also* $\mathfrak{G}$ *be an algebra of relative convolutions induced by* $\mathfrak{g}$ *on* S *and let* $\widehat{\mathfrak{G}}$ *be a group convolution algebra on* G. *Then* $\mathfrak{G}$ *is a linear representation of* $\widehat{\mathfrak{G}}$.

Remark 3.4. Due to Theorem 3.3 we can give an alternative definition of a relative convolution algebra, namely, *an operator algebra* $\mathfrak{G}$ *is a relative convolution algebra induced by a Lie algebra* $\mathfrak{g}$, *if all representations of* $\mathfrak{G}$ *are subrepresentations of the convolution algebra on group* $\mathrm{Exp}\,\mathfrak{g}$.

Theorem 3.3 is the main tool for applications of harmonic analysis to every problem where relative convolutions occur. Particulary, it gives us an easy access to many conclusions obtained by direct calculations. Now we illustrate this by applications to two-sided convolutions from Example 2.7.

Corollary 3.5. [36] An algebra of two-sided convolutions on the Heisenberg group $\mathbb{H}^n$ is a representation of the group convolution algebra on $\mathbb{H}^{2n}$.

[4]Relative convolutions obviously include usual group convolutions and, on the other hand, due to Theorem 3.3 they may be treated just as representations of group convolutions. Thus they are not *generalized* convolutions but simply *relative* convolutions.

Explicit descriptions of the established representation depend from properties of relative convolutions kernels. In the next paper will be shown that for kernels from $L_1(\mathbb{R}^N \times \mathbb{R}^N)$ there is a one-to-one correspondence between representations of two-sided convolutions on $\mathbb{R}^N$ and usual one-sided convolution on the $\operatorname{Exp}\operatorname{Ess}(\mathfrak{g}\times\mathfrak{g}, \mathbb{R}^N)$ (for the Heisenberg group it was calculated in [30]). If kernels have a symmetry group then some representations of $\operatorname{Exp}\operatorname{Ess}(\mathfrak{g}\times\mathfrak{g}, \mathbb{R}^N)$ vanish, see, for example case of two-sided convolutions on $\mathbb{H}^n$ with homogeneous kernels (2.7) in [53].

There is another simple but important corollary of Theorem 3.3

Corollary 3.6. *Let $\psi : \mathfrak{g} \to \mathfrak{g}$ be an automorphism of $\mathfrak{g}$ as a Lie algebra. Let $\Psi : \mathfrak{G} \to \mathfrak{G}$ be a mapping defined by the rule $\Psi : K \mapsto \Phi(K)$, where K is a relative convolution with a kernel $k(x)$, $\Psi(K)$ is a relative convolution with the kernel $k(\psi x)\, J^{1/2}\psi(x)$ and $J\psi(x)$ is the Jacobian of ψ at the point x. Then Ψ is an automorphism of algebra $\mathfrak{G}$.*

Proof. The key point of the proof is the invariance of (3.2) under Ψ (the rest is almost evident). This invariance follows from a simple change of variables in the integral (2.4):

$$\begin{aligned}
\Psi[k_1 * k_2](x) &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} k_2(y)\, k_1(\mathrm{CH}\,[-y, \psi(x)])\, J^{1/2}\psi(x)\, dy \\
&= (2\pi)^{-N/2} \int_{\mathbb{R}^N} k_2(y)\, k_1(\mathrm{CH}\,[-\psi(\psi^{-1}(y)), \psi(x)])\, J^{1/2}\psi(x)\, dy \\
&= (2\pi)^{-N/2} \int_{\mathbb{R}^N} k_2(\psi(y'))\, k_1(\mathrm{CH}\,[-\psi(y'), \psi(x)])\, J^{1/2}\psi(x)\, dy \\
&\overset{(*)}{=} (2\pi)^{-N/2} \int_{\mathbb{R}^N} k_2(\psi(y'))\, k_1(\psi(\mathrm{CH}\,[-y', x]))\, J^{1/2}\psi(x)\, dy \\
&= (2\pi)^{-N/2} \int_{\mathbb{R}^N} k_2(\psi(y'))\, J^{1/2}\psi(y')\, k_1(\psi(\mathrm{CH}\,[-y', x])) \\
&\qquad \times J^{1/2}\psi(y' * x)\, J^{-1}\psi(y')\, dy \\
&= (2\pi)^{-N/2} \int_{\mathbb{R}^N} \Psi k_2(y')\, \Psi k_1(\mathrm{CH}\,[-y', x])\, dy' \\
&= [\Psi k_2 * \Psi k_1](x),
\end{aligned}$$

where $y' = \psi^{-1}(y),\ y = \psi(y')$. Here transform $(*)$ is possible due to the automorphism property of ψ. We also employ the identity

$$J^{1/2}\psi(y')\, J^{1/2}\psi(y' * x)\, J^{-1}\psi(y') = J^{1/2}\psi(x),$$

which follows from the chain rule. □

Next Lemma evidently follows from Definition 2.3 and Theorem 3.3.

Lemma 3.7. *Let an algebra $\widetilde{\mathfrak{G}}$ be a representation of an algebra $\mathfrak{G}$ of relative convolutions induced by a Lie algebra $\mathfrak{g}$. Then $\widetilde{\mathfrak{G}}$ is also the algebra of relative convolutions induced by $\mathfrak{g}$.*

Otherwise, algebras of relative convolutions induced by a Lie algebra $\mathfrak{g}$ form a closed sub-category $\mathcal{G}$ of the category of all operator algebras with morphisms defined by representations (up to unitary equivalence) of algebras. The universal repelling object of the category $\mathcal{G}$ is the algebra $\mathfrak{G}$ of group convolutions on $\operatorname{Exp}\mathfrak{g}$.

Besides a convolution-like calculus, there is another well developed type of calculus, namely, the calculus of pseudodifferential operators (PDO), which is highly useful in analysis. Let us remind, a PDO $\operatorname{Op} a(x,\xi)$ [27, 48, 50, 54] with the *Weyl symbol* $a(x,\xi)$ is defined by the formula:

$$[\operatorname{Op} a](x,\xi)u(y) = \int_{\mathbb{R}^n\times\mathbb{R}^n} a(\frac{x+y}{2},\xi)\, e^{i\langle y-x,\xi\rangle} u(x)\, dx\, d\xi. \tag{3.3}$$

It was shown that relative convolutions are PDOs for many types studed before (for example, families of convolutions [51, Proposition 1.1] and meta-Heisenberg group [22]). But PDOs itself are relative convolutions induced by the Heisenberg group (see Example 4.1). Thus if we consider morphisms at categories of relative convolution algebras only up to smooth operators, then we have[5]

Theorem 3.8. *The category $\mathcal{G}$ of relative convolution algebras induced by a Lie algebra $\mathfrak{g}$, $\dim \mathfrak{g} = N$ is (up to smooth operators) a sub-category of the category $\mathcal{H}^N$ of relative convolution algebras induced by the Heisenberg group $\mathbb{H}^N$.*

Proof. It is well known [51, Proposition 1.1], that group convolutions on $\operatorname{Exp}\mathfrak{g}$ are (up to smooth operators, at least) PDOs on $\mathbb{R}^N$ and thus the algebra of convolutions belongs to $\mathcal{H}^N$. The rest of the assertion is given by Theorem 3.3 and Lemma 3.7. □

Of course, there is still a huge distance between Theorem 3.8 and the real PDO-like calculus of relative convolutions.

Calculus of PDO is a representation [29] of a group convolution calculus on the Heisenberg group (the simplest nilpotent Lie group). Thus Theorem 3.8 establishes connection between relative convolutions and convolutions on the Heisenberg group. Therefore it is not surprising, that for convolutions on nilpotent Lie groups this connection has a very simple form (see [35, Theorem 1]).

4. Applications to Complex and Hypercomplex Analysis

In this and the next Sections we would like to introduce a series of essential Examples, which show possible applications[6] of relative convolutions. Due to the wide spectrum of Examples only brief descriptions will be given here. We are going to return to these subjects in the future papers in this series after an appropriate development of general theory of relative convolutions.

As was pointed by Hörmander in [1], formula (2.4) is closely connected with a partial differential equation with operator coefficients. Thus it is not surprizing, that most Examples within this Section are touching some spaces of holomorphic functions, which are solutions to corresponding equations. In Example 4.7 this connection will be used directly.

Example 4.1. Let us consider a combination of Examples 2.5 and 2.6. Namely, let $\mathfrak{g}$ has a frame consisting from vector fields

$$X_j = y_j,\ X_j^e = \frac{1}{i}\frac{\partial}{\partial y_j},\ 1 \le j \le N, \tag{4.1}$$

[5] "The Heisenberg group ... is basic to this paper (and much of the rest of the word)" [29].

[6] "The most interesting aspect of the ... theory has to do with the application of this machinery to concrete examples" [25].

which operate on $S=\mathbb{R}^N$ by the obvious way. Note, that these vector fields have exactly the same commutators (2.12) as left (right) fields from (2.10) (or (2.11)) if we put $X_0^{l(r)}=iI$. Then an operator of relative convolution with the kernel $\widehat{k}(x,\xi),\ x,\xi\in\mathbb{R}^N$ has form (see [52, § 1.3])

$$\begin{aligned}[Kf](y) &= (2\pi)^{-N}\int_{\mathbb{R}^{2N}}\widehat{k}(x,\xi)\,e^{i(\sum_1^N x_jy_j-\sum_1^N \xi\frac{\partial}{i\partial y_j})}f(y)\,dx\,d\xi\\ &= (2\pi)^{-N}\int_{\mathbb{R}^{2N}}k(\frac{x+y}{2},\xi)\,e^{i(y-x)\xi}f(x)\,dx\,d\xi,\end{aligned}$$

i. e. it exactly defines the *Weyl functional calculus* (or the Weyl quantization) [27, 48, 50, 54]. This forms a very important tool for the theory of differential equations and quantum mechanics.

Our definition of relative convolution operators obviously generalizes the Weyl functional calculus from the case of the Heisenberg group to an arbitrary exponential Lie group. A deep investigation of the role of the Heisenberg group (and its different representations) at PDO calculus and harmonic analysis on real line may be found at [28, 29]. Unfortunately, usual definitions of PDO symbol classes S^m destroy the natural symmetry between x and ξ and this restricts applications of harmonic analysis to the (standard) PDO theory. Particulary, $S^m(\mathbb{R}^{2n})$ is not invariant under all symplectic morphisms of $\mathbb{R}^{2n}$, which are induced by automorphisms of the Heisenberg group (see Corollary 3.6).

Example 4.2. Now we follow the paper [12] footsteps but only in the original context of complex analysis (see also an elegant survey in [49, Chap. XII]). Let $\mathbb{U}^n$ be an *upper half-space*

$$\mathbb{U}^n=\{z\in\mathbb{C}^{n+1}\ \mid\ \operatorname{Im} z_{n+1}>\sum_{j+1}^n|z_j|^2\},$$

which is a domain of holomorphy of functions of $n+1$ complex variables. Its boundary

$$b\mathbb{U}^n=\{z\in\mathbb{C}^{n+1}\ \mid\ \operatorname{Im} z_{n+1}=\sum_{j+1}^n|z_j|^2\}$$

may be naturally identified with the Heisenberg group $\mathbb{H}^n$. One can introduce the *Szegö projector* R as the orthogonal projection of $L_2(\mathbb{H}^n)$ onto its subspace $H_2(\mathbb{H}^n)$ (the *Hardy space*) of boundary values of holomorphic functions on the upper half-space $\mathbb{U}^n$. Then a *Toeplitz operator* [12] on $\mathbb{H}^n$ with the pre-symbol Q is an operator of the form $T_Q=RQR$ where $Q:L_2(\mathbb{H}^n)\to L_2(\mathbb{H}^n)$ is a pseudodifferential operator. Obviously $T_Q:H_2(\mathbb{H}^n)\to H_2(\mathbb{H}^n)$. The invariance of the *tangential Cauchy–Riemann equations* under right shifts of $\mathbb{H}^n$ implies that the Szegö projector can be realized as a (left) convolution operator on $\mathbb{H}^n$ [23] (see also Corollary 5.14). Thus the algebra of the Toeplitz operators on $\mathbb{H}^n$ can be naturally imbedded into the algebra of (pseudodifferential) operators generated by left group convolutions on $\mathbb{H}^n$ and PDO.

First, let us consider a case of a pre-symbol Q taken from usual Euclidean convolutions on $\mathbb{H}^n\cong\mathbb{R}^{2n+1}$. Left convolutions on $\mathbb{H}^n$ are generated by vector fields

X_j^l from (2.10) and Euclidean convolutions are induced by fields (see Example 2.5)

$$X_j^e = \frac{1}{i}\frac{\partial}{\partial y_j}, \ 1 \le j \le N.$$

But two frames of vector fields $\{X_j^l, X_j^e\}$ and $\{X_j^l, X_j^r\}$ define just the same action on $\mathbb{H}^n$. Therefore we have an embedding of the Toeplitz operators with Euclidean convolution pre-symbols to the algebra of two-sided convolutions on $\mathbb{H}^n$ from Example 2.7.

If we now allow Q to be a general PDO from Example 4.1, then we should consider a joint operation of vector fields $X_j,\ X_j^e$ from (4.1) and X_j^l from (2.10). Again one can pass to equivalent frame defined by $X_j,\ X_j^l,\ X_j^r$. The algebra of relative convolutions defined by the last frame is the algebra of operators generated by two-sided convolutions on $\mathbb{H}^n$ and operators of multiplication by functions, which form a meta-Heisenberg group [22]. Such algebras for continuous multipliers were studied in [35, 34].

There is our conclusion[7]:

Proposition 4.3. *The algebra of the Toeplitz operators with PDO pre-symbols is naturally imbedded into the algebra of relative convolutions generated by two-sided convolutions on $\mathbb{H}^n$ and operators of multiplication by functions.*

For the first time the algebra of the Toeplitz operators with two-sided convolution pre-symbols was studied at [33].

We are going to consider another problem from complex analysis.

Example 4.4. Let $L_2(\mathbb{C}^n, d\mu_n)$ be a space of all square-integrable functions on $\mathbb{C}^n$ with respect to the Gaussian measure

$$d\mu_n(z) = \pi^{-n} e^{-z\cdot\overline{z}} dv(z),$$

where $dv(z) = dx\,dy$ is the usual Euclidean volume measure on $\mathbb{C}^n = \mathbb{R}^{2n}$. Denote by P_n the orthogonal Bargmann projector of $L_2(\mathbb{C}^n, d\mu_n)$ onto the *Segal–Bargmann* or *Fock space* $F_2(\mathbb{C}^n)$, namely, the subspace of $L_2(\mathbb{C}^n, d\mu_n)$ consisting of all entire functions. The Fock space $F_2(\mathbb{C}^n)$ was introduced by Fock [20] to give an alternative representation of the Heisenberg group in quantum mechanics. The rigorous theory of $F_2(\mathbb{C}^n)$ was developed by Bargmann [2] and Segal [47]. Such a theory is closely connected with representations of the Heisenberg group (see also [21, 25, 29]), but studies of the Bargmann projector and the associated Toeplitz operators are usually based on the Hilbert spaces technique[8]. There is a strong reason for this: the Bargmann projection P_n is not a *group* convolution for any group. However, it is possible to consider P_n as a *relative* convolution.

Let us consider the group of Euclidean shifts $a : z \mapsto z + a$ of $\mathbb{C}^n$. To make unitary operators on $L_2(\mathbb{C}^n, d\mu)$ from the shifts we should multiply by the special weight function:

$$a : f(z) \mapsto f(z+a) e^{-z\bar{a} - a\bar{a}/2}. \tag{4.2}$$

It is obvious, that (4.2) defines a unitary representation [29] of the $(2n+1)$-dimensional Heisenberg group on $L_2(\mathbb{C}^n, d\mu)$, which preserves the Fock space $F_2(\mathbb{C}^n)$.

[7] This is an answer to the reasonable question of E. Stein: "Does the algebra of two-sided convolutions contain at least one interesting operator?"

[8] Or, at least, do not use harmonic analysis directly.

Thereafter all operators (4.2) should commute with P_n. Unitary operators (4.2) have such infinitesimal generators

$$i\sum_{k=1}^{n}\left(a_j'(\frac{\partial}{\partial z_j'}-z_j'-iz_j'')+a_j''(\frac{\partial}{\partial z_j''}-z_j''+iz_j')\right),$$

where $a=(a_1,\ldots,a_n), z=(z_1,\ldots,z_n)\in\mathbb{C}^n$ and $a_j=(a_j',a_j''), z=(z_j',z_j'')\in\mathbb{R}^2$. This linear space of generators has the frame

$$A_j^{f\prime}=\frac{1}{i}\left(\frac{\partial}{\partial z_j'}-z_j'-iz_j''\right),\ A^{f\prime\prime}=\frac{1}{i}\left(\frac{\partial}{\partial z_j''}-z_j''+iz_j'\right). \tag{4.3}$$

Operators (4.3) still should commute with the Bargmann projector P_n and we can expect that P_n should be a relative convolution with respect to operators

$$X_j^{f\prime}=\frac{1}{i}\left(\frac{\partial}{\partial z_j'}-z_j'+iz_j''\right),\ X^{f\prime\prime}=\frac{1}{i}\left(\frac{\partial}{\partial z_j''}-z_j''-iz_j'\right), \tag{4.4}$$

which commute with all operators (4.3) and together with operator $2I$ form a self-adjoint representation of $\mathfrak{h}_n$. Indeed, we have

Proposition 4.5. *The Bargmann projector is a relative convolution induced by the Weyl-Heisenberg Lie algebra $\mathfrak{h}_n$, which have an operation on $\mathbb{C}^n$ defined by* (4.4). *Its kernel $b(t,\zeta),\ t\in\mathbb{R},\ \zeta\in\mathbb{C}^n$ is defined by the formula:*

$$\widehat{b}(t,\zeta)=2^{n+1/2}e^{-1}e^{-(t^2+\zeta\bar\zeta/2)}.$$

Proof. One can make an easy exercise with integral transforms:

$$\begin{aligned}
[P_nf](z) &= (2\pi)^{-n-1/2}\int_{\mathbb{R}}\int_{\mathbb{C}^n}2^{n+1/2}\,e^{-(t^2+1+\zeta\bar\zeta)/2}\\
&\qquad\times e^{-i(t\cdot 2I+\sum_{k=1}^n(\zeta_j'X_j^{f\prime}+\zeta_j''X_j^{f\prime\prime}))}\,f(z)\,dt\,d\zeta\\
&= \pi^{-n-1/2}\int_{\mathbb{R}}e^{-(t^2+1+2it)}\,dt\int_{\mathbb{C}^n}e^{-(\zeta\bar\zeta)/2}\,e^{-i\sum_{k=1}^n(\zeta_j'X_j^{f\prime}+\zeta_j''X_j^{f\prime\prime})}\,f(z)\,d\zeta\\
&= \pi^{-n}\int_{\mathbb{C}^n}e^{-\zeta\bar\zeta/2}\,e^{-i\sum_{k=1}^n(\zeta_j'X_j^{f\prime}+\zeta_j''X_j^{f\prime\prime})}\,f(z)\,d\zeta\\
&= \pi^{-n}\int_{\mathbb{C}^n}e^{-\zeta\bar\zeta/2}\,e^{-\sum_{k=1}^n(\zeta_j'\frac{\partial}{\partial z_j'}+\zeta_j''\frac{\partial}{\partial z_j''}-\zeta_j\bar z_j)}\,f(z)\,d\zeta\\
&= \pi^{-n}\int_{\mathbb{C}^n}e^{-\zeta\bar\zeta/2}\,e^{-\zeta\bar\zeta/2}\,e^{\zeta\bar z}\,e^{-\sum_{k=1}^n(\zeta_j'\frac{\partial}{\partial z_j'}+\zeta_j''\frac{\partial}{\partial z_j''})}\,f(z)\,d\zeta\\
&= \pi^{-n}\int_{\mathbb{C}^n}e^{(\bar z-\bar\zeta)\zeta}\,f(z-\zeta)\,d\zeta\\
&= \pi^{-n}\int_{\mathbb{C}^n}e^{\bar w(z-w)}\,f(w)\,dw, \qquad (4.5)
\end{aligned}$$

where $\zeta=(\zeta',\zeta'')\in\mathbb{R}^n\times\mathbb{R}^n,\ w=z-\zeta$. Formula (4.5) is the well known expression for the Bargmann projector [2]. □

See also Corollary 5.12 for an alternative proof.

Following to papers [8, 9, 13] we now consider the Toeplitz operator of the form $T_a=P_na(z)I$, where $a(z)I$ is an operator of multiplication by a function $a(z)$.

For the previous reasons we can handle them as relative convolutions generated by operators X_j^f from (4.4) and operators $X_j = z_j I$. We can easily describe all non-zero commutators:

$$[X_j^{f\prime}, X_j^{f\prime\prime}] = 2iI,\ [X_j^{f\prime}, X_j^{\prime}] = iI,\ [X_j^{f\prime\prime}, X_j^{\prime\prime}] = iI. \tag{4.6}$$

So they form a $(4n+1)$-dimensional nilpotent step 2 Lie algebra. Particulary subalgebra spanned on the vectors $X_j^{f\prime}$, $X_j^{f\prime\prime}$ and iI is isomorphic[9] to our constant companion $\mathfrak{h}_n$.

Proposition 4.6. *There is a natural embedding of the Toeplitz operator algebra on the Fock space into the algebra of relative convolutions induced by the Lie algebra with commutation relations* (4.6)*.*

We will continue our discussion of Toeplitz operators on an "abstract nonsense" [29] level at Section 5.

Example 4.7. Let now X_j be generators of the Clifford algebra $\mathcal{C}\ell(0)$n (we use book [14] as a standard reference within this Example). This means that the following *anti-commutation* relations hold (compare with (2.12)):

$$\{X_i, X_j\} := X_i X_j + X_j X_i = -2\delta_{ij} X_0, \tag{4.7}$$

where $X_0 = I$. Function $f : \mathbb{R}^n \to \mathcal{C}\ell(0)n$ is called *monogenic* if it satisfies the *Dirac equation*

$$Df := \frac{\partial f(y)}{\partial y_0} - \sum_{j=1}^{n} X_j \frac{\partial f(y)}{\partial y_j} = 0 \quad \text{or} \quad \frac{\partial f(y)}{\partial y_0} = \sum_{j=1}^{n} X_j \frac{\partial f(y)}{\partial y_j}. \tag{4.8}$$

The success of Clifford analysis is mainly explained because the Dirac operator (4.8) factorizes the Laplace operator $\Delta = \sum_0^n \frac{\partial^2}{\partial x_j^2}$.

Hörmander's remark from paper [1] gives us by the fundamental solution to the Dirac equation in the form

$$\begin{aligned} K(y) &= \mathcal{F}^{\eta\to y} e^{-iy_0 \sum_{j=1}^{n} \eta_j X_j} \\ &= \int_{\mathbb{R}^n} e^{i\sum_{j=1}^{n} y_j \eta_j}\, e^{-iy_0 \sum_{j=1}^{n} \eta_j X_j}\, d\eta. \end{aligned}$$

("Simply take the Fourier transform with respect to the spatial variables, and solve the equation in y_0" [1]). Otherwise, any solution $f(y)$ to (4.8) is given by a convolution of some function $\widetilde{f}(y)$ on $\mathbb{R}^{n-1}$ and the fundamental solution $K(y)$. On the

[9] This explains why "a critical ingredient in our analysis is an averaging operation over the Segal–Bargmann representation of the Heisenberg group" [9].

contrary, a convolution $K(y)$ with any function is a solution to (4.8). We have:

$$\begin{aligned} [K*f](y) &= \int_{\mathbb{R}^n} K(y-t)\, f(t)\, dt \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n (y_j-t_j)\eta_j}\, e^{-iy_0\sum_{j=1}^n \eta_j X_j}\, d\eta f(t)\, dt \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\sum_{j=1}^n t_j\eta_j}\, e^{-i\sum_{j=1}^n \eta_j(y_0X_j-y_jX_0)}\, d\eta f(t)\, dt \\ &= \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n \eta_j(y_0X_j-y_jX_0)} \int_{\mathbb{R}^n} e^{i\sum_{j=1}^n t_j\eta_j}\, f(t)\, dt\, d\eta \\ &= \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n \eta_j(y_0X_j-y_jX_0)}\, \hat{f}(-\eta)\, d\eta. \end{aligned} \tag{4.9}$$

Equation (4.9) defines relative convolution (2.4) with the kernel $f(-y)$ and the Lie algebra of vector fields

$$\{y_0X_j - y_jX_0\},\ 1 \le j \le n. \tag{4.10}$$

Thus, at least formally, any solution to the Dirac equation (4.8) can be written as a function (see Remark 2.4) of $n-1$ monomials (4.10):

$$\breve{f}(y_0, y_1, \ldots, y_n) = f(y_0X_1 - y_1X_0, y_0X_2 - y_2X_0, \ldots, y_0X_n - y_nX_0). \tag{4.11}$$

Another significant remark: if we fix the value $y_0 = 0$ in (4.11) we easily obtain: $\breve{f}(0, y_1, \ldots, y_n) = f(-y_1X_0, -y_2X_0, \ldots, -y_nX_0) = f(-y_1, -y_2, \ldots, -y_n)$. Thus we may consider the function $\breve{f}(y_0, y_1, \ldots y_n)$ of $n+1$ variables as *analytical expansion* for the function $f(y_1, \ldots, y_n)$ of n variables (compare with [41]).

Using the power series decomposition for the exponent one can see that formula (4.9) defines the permutational (symmetric) product of monomials (4.10). The significant role of such monomials and functions generated by them in Clifford analysis was recently discovered by Laville [40] and Malonek [42]. But during our consideration we used only the commutation relation $[X_0, X_j] = 0$ and never used the anti-commutation relations (4.7). Thus formula (4.9) *is true and may be useful without Clifford analysis* as well.

Proposition 4.8. *Any solution to equation* (4.8)*, where X_j are arbitrary self-adjoint operators, is given as arbitrary function of n monomials* (4.10) *by the formula* (4.9)*.*

It is possible also to introduce the notion of the *differentiability* [42] for solutions to (4.8), namely, an increment of any solution to (4.8) may be approximated up to infinitesimals of the second order by a linear function of monomials (4.10).

Due to physical application we will consider equation

$$\frac{\partial f}{\partial y_0} = (\sum_{j=1}^n X_j \frac{\partial}{\partial y_j} + M)f, \tag{4.12}$$

where X_j are arbitrary self-adjoint operators and M is a bounded operator commuting with all X_j.

Remark 4.9. Then X_j are generators (4.7) of the Clifford algebra and $M = M_\alpha$ is an operator of multiplication from the *right*-hand side by the Clifford number

α, differential operator (4.12) factorizes the Helmholtz operator $\Delta + M_{\alpha^2}$. Equation (4.12) is known in quantum mechanics as the *Dirac equation for a particle with a non-zero rest mass* [3, §20], [11, §6.3] and [38].

Simple modification of the previous calculations gives us the following result

Proposition 4.10. *Any solution to equation* (4.12)*, where X_j are arbitrary self-adjoint operators and M commutes with them, is given by the formula*

$$e^{y_0 M} \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^{n} \eta_j (y_0 X_j - y_j X_0)} \widehat{f}(-\eta)\, d\eta,$$

where f is an arbitrary function on $\mathbb{R}^{n-1}$.

Remark 4.11. Connection between relative convolutions and Clifford analysis is two-sided. Not only relative convolutions are helpful in Clifford analysis but also Clifford analysis may be used for developing the relative convolution technique. Indeed, we have defined a relative convolution as a function of operators X_j representing a Lie algebra $\mathfrak{g}$. To do this we have used the Weyl function calculus from [1]. Meanwhile, for a pair of self-adjoint operators $X_1,\ X_2$ the alternative Riesz calculus [45, Chap. XI] is given by the formula

$$f(X_1 + iX_2) = \int_{\Gamma} f(\tau)(\tau I - (X_1 + iX_2))^{-1}\, d\tau. \tag{4.13}$$

As shown in [1], these two calculuses are essentially the same in the case of a pair of bounded operators. To extend the Riesz calculus for arbitrary n-tuples of bounded operators $\{X_j\}$ it seems natural to use Clifford analysis (see, for example, [14]), which is an analogy to one-dimensional complex analysis. Then one can define a function of arbitrary n-tuple of bounded self-adjoint operators $\{X_j\}$ in such a way (compare with (4.13)):

$$f(X_1, X_2, \ldots, X_n) = \int_{\Gamma} f(\tau_1, \tau_2, \ldots, \tau_n)\, K(\tau_1 I - X_1, \tau_2 I - X_2, \ldots, \tau_n I - X_n)\, d\tau,$$

where $K(\tau_1, \tau_2, \ldots, \tau_n)$ is the Cauchy kernel fron the Clifford analysis [14] .

5. Coherent States

This Section is a bridge between the previous and the next ones. Here we will give a new glance on some constructions of Examples 4.2 and 4.4. We also provide a foundation for the further investigation in Section 6.

5.1. **General Consideration.**

Coherent states are a useful tool in quantum theory and have a lot of essentially different definitions [44]. Particulary, they were described by Berezin in [4, 6, 25] concerning so-called covariant and contravariant (or Wick and anti-Wick) symbols of operators (quantization).

Definition 5.1. They say that the Hilbert space H has a system of coherent states $\{f_\alpha\}, \alpha \in G$ if for any $f \in H$

$$\langle f, f \rangle = \int_G |\langle f, f_\alpha \rangle|^2\, d\mu. \tag{5.1}$$

This definition does not take in account that within coherent states a group structure frequently occurs and is always useful [44]. For example, the original consideration of Berezin is connected with the Fock space, where coherent states are functions $e^{za-a\bar{a}/2}$. The representation of the Heisenberg group on the Fock space was exploited in Example 4.4. Another type of coherent states with a group structure is given by the vacuum vector and operators of creation and annihilation, which represents group $\mathbb{Z}$. Thus we would like to give an alternative definition.

Definition 5.2. We will say that the Hilbert space H has a system of coherent states $\{f_g\},\ g \in G$ if

(1) There is a representation $T : g \mapsto T_g$ of the group G by unitary operators T_g on H.
(2) There is a vector $f_0 \in H$ that for $f_g = T_g f_0$ and arbitrary $f \in H$ we have

$$\langle f, f\rangle = \int_G |\langle f, f_g\rangle|^2 \, d\mu, \tag{5.2}$$

where integration is taken over the Haar measure $d\mu$ on G.

Because this construction independently arose in different contexts, vector f_0 has many various names: the *vacuum vector*, the *ground state*, the *mother wavelet* etc. Modifications of definition 5.2 for other cases are discussed [44, § 2.1]. Equation (5.2) implies, that vector f_0 is a cyclic vector of the representation G on H.

Lemma 5.3. *Let T is a unitary representation of a group G in a Hilbert space H. Then there exists such $f_0 \in H$ that equality* (5.2) *holds. Moreover, if the representation T is irreducible, then one can take an arbitrary non-zero vector of H (up to a scalar factor) as the vacuum vector.*

Proof. Let us fix some Haar measure $d\mu$ on G. (Different Haar measures are different on a scalar factor). If the representation T is irreducible, then an arbitrary vector $f \in H$ is cyclic and we may put $f_0 = c^{-1/2} f$, where

$$c = \frac{\int_G |\langle f, T_g f\rangle|^2 \, d\mu(g)}{\langle f, f\rangle}.$$

It is easy to verify, that for the f_0 equality (5.2) holds [44, § 2.3].

Let now T is an arbitrary representation, then we can decompose [32, § 8.4] it onto a direct integral [16, § 10] of irreducible representations $T = \int_Y T_\alpha \, d\alpha$ on the space $H = \int_Y H_\alpha \, d\alpha$. Again we can take an arbitrary vector $f = \int_Y f_\alpha \, d\alpha \in H = \int_Y H_\alpha \, d\alpha$, such that f_α are non-zero almost everywhere, and put $f_0 = \int_Y f_\alpha c_\alpha^{-1/2} d\alpha$, where

$$c_\alpha = \frac{\int_G \left|\langle f_\alpha, T_g f_\alpha\rangle_\alpha\right|^2 d\mu(g)}{\langle f_\alpha, f_\alpha\rangle_\alpha}.$$

In view of $\|f\| = \int_Y \|f_\alpha\| \, d\alpha$, the proof is complete. □

By the way, a polarization of (5.2) gives us the equality

$$\langle f_1, f_2\rangle = \int_G \langle f_1, f_g\rangle \, \overline{\langle f_2, f_g\rangle} \, d\mu. \tag{5.3}$$

Thus we have an isometrical embedding $E : H \to L_2(G, d\mu)$ defined by the formula

$$E : f \mapsto f(g) = \langle f, f_g\rangle = \langle f, T_g f_0\rangle = \left\langle T_g^* f, f_0\right\rangle = \left\langle T_{g^{-1}} f, f_0\right\rangle. \tag{5.4}$$

We will consider $L_2(G, d\mu)$ both as a linear space of functions and as an operator algebra with respect to the left and right group convolution operations:

$$[f_1 * f_2]_l(h) = \int_G f_1(g) f_2(g^{-1}h)\, d\mu(g), \tag{5.5}$$

$$[f_1 * f_2]_r(h) = \int_G f_1(g) f_2(hg)\, d\mu(g). \tag{5.6}$$

For a simplicity we will assume, that G is unimodular (the left and the right Haar measures on G coincide) and that $L_2(G, d\mu)$ is closed under the group convolution. Thus the construction under consideration may be regarded as a natural embedding of the linear space H to the operator algebra $\mathcal{B}(H)$.

Let $H_2(G, d\mu) \subset L_2(G, d\mu)$ will denote the image of H under embedding E. It is clear, that $H_2(G, d\mu)$ is a liner subspace of $L_2(G, d\mu)$, which does not coincide with the whole $L_2(G, d\mu)$ in general. One can see, that

Lemma 5.4. *Space $H_2(G, d\mu)$ is invariant under left shifts on G.*

Proof. Indeed, for every $f(g) \in H_2(G, d\mu)$ the function

$$f(h^{-1}g) = \left\langle f, T_{h^{-1}g} f_0 \right\rangle = \langle f, T_{h^{-1}} T_g f_0 \rangle = \langle T_h f, T_g f_0 \rangle = [T_h f](g)$$

also belongs to $H_2(G, d\mu)$. □

If $P : L_2(G, d\mu) \to H_2(G, d\mu)$ the orthogonal projector on $H_2(G, d\mu)$, then due to Lemma 5.4 it should commute with all left shifts and thus we get immediately

Corollary 5.5. *Projector $P : L_2(G, d\mu) \to H_2(G, d\mu)$ is a right convolution on G.*

The following Lemma characterizes *linear subspaces* of $L_2(G, d\mu)$ invariant under shifts in the term of *convolution algebra* $L_2(G, d\mu)$ and seems to be of the separate interest.

Lemma 5.6. *A closed linear subspace H of $L_2(G, d\mu)$ is invariant under left (right) shifts if and only if H is a left (right) ideal of the right group convolution algebra $L_2(G, d\mu)$.*

A closed linear subspace H of $L_2(G, d\mu)$ is invariant under left (right) shifts if and only if H is a right (left) ideal of the left group convolution algebra $L_2(G, d\mu)$.

Proof. Of course we consider only the "right-invariance and right-convolution" case. Then the other three cases are analogous. Let H be a closed linear subspace of $L_2(G, d\mu)$ invariant under right shifts and $k(g) \in H$. We will show the inclusion

$$[f * k]_r(h) = \int_G f(g) k(hg)\, d\mu(g) \in H, \tag{5.7}$$

for any $f \in L_2(G, d\mu)$. Indeed, we can treat integral (5.7) as a limit of sums

$$\sum_{j=1}^{N} f(g_j) k(hg_j) \Delta_j. \tag{5.8}$$

But the last sum is simply a linear combination of vectors $k(hg_j) \in H$ (by the invariance of H) with coefficients $f(g_j)$. Therefore sum (5.8) belongs to H and this is true for integral (5.7) by the closeness of H.

Otherwise, let H be a right ideal in the group convolution algebra $L_2(G, d\mu)$ and let $\phi_j(g) \in L_2(G, d\mu)$ be an approximate unit of the algebra [16, § 13.2], i. e. for any $f \in L_2(G, d\mu)$ we have

$$[\phi_j * f]_r(h) = \int_G \phi_j(g) f(hg)\, d\mu(g) \to f(h), \text{ when } j \to \infty.$$

Then for $k(g) \in H$ and for any $h' \in G$ the right convolution

$$[\phi_j * k]_r(hh') = \int_G \phi_j(g) k(hh'g)\, d\mu(g) = \int_G \phi_j(h'^{-1}g') k(hg')\, d\mu(g'),\ \ g' = h'g,$$

from the first expression is tensing to $k(hh')$ and from the second one belongs to H (as a right ideal). Again the closeness of H implies $k(hh') \in H$ that proves the assertion. □

Lemma 5.7. (The reproducing property) For any $f(g) \in H_2(G, d\mu)$ we have

$$[f * f_0]_l(g) = f(g) \tag{5.9}$$

$$[\bar{f}_0 * f]_r(g) = f(g), \tag{5.10}$$

where $f_0(g) = \langle f_0, T_g f_0 \rangle$ is the function corresponding to the vacuum vector $f_0 \in H$.

Proof. We again check only the left case and this is just a simple calculation:

$$\begin{aligned}
[f * f_0]_l(h) &= \int_G f(g)\, f_0(g^{-1}h)\, d\mu(g)\\
&= \int_G f(g) \left\langle f_0, T_{g^{-1}h} f_0 \right\rangle\, d\mu(g)\\
&= \int_G \langle f, T_g f_0 \rangle \left\langle f_0, T_{g^{-1}} T_h f_0 \right\rangle\, d\mu(g)\\
&= \int_G \langle f, T_g f_0 \rangle \langle T_g f_0, T_h f_0 \rangle\, d\mu(g)\\
&= \int_G \langle f, T_g f_0 \rangle\, \overline{\langle T_h f_0, T_g f_0 \rangle}\, d\mu(g)\\
&\overset{(*)}{=} \langle f, T_h f_0 \rangle\\
&= f(h).
\end{aligned}$$

Here transformation $(*)$ is based on (5.3) and we have used the unitary property of the representation T. □

The following general Theorem easily follows from the previous Lemmas

Theorem 5.8. *The orthogonal projector $P : L_2(G, d\mu) \to H_2(G, d\mu)$ is a right convolution on G with the kernel $\bar{f}_0(g)$ defined by the vacuum vector.*

To put in the Archimedes-like words, *let me a representation of group G on H with a cyclic vector f_0 and I will construct the projector $P : L_2(G, d\mu) \to H_2(G, d\mu) \cong H$.*

Proof. Let P be the operator of right convolution (5.6) with the kernel $\bar{f}_0(g)$. By the Lemma 5.4 $H_2(G, d\mu)$ is an invariant linear subspace of $L_2(G, d\mu)$. Thus by Lemma 5.6 it is an ideal under convolution operators. Therefore the convolution operator P with the kernel $\bar{f}_0(g)$ from $H_2(G, d\mu)$ has an image belonging to $H_2(G, d\mu)$. But by Lemma 5.7 $P = I$ on $H_2(G, d\mu)$, so $P^2 = P$ on $L_2(G, d\mu)$, i. e. P is a projector on $H_2(G, d\mu)$.

It is easy to see, that $f_0(g)$ has the property $f_0(g) = \bar{f}_0(-g)$, thus $P^* = P$, i. e. P is orthogonal. It may be shown also in such a manner. Let $f(g) \in L_2(G, d\mu)$ be orthogonal to all functions from $H_2(G, d\mu)$. Particulary $f(g)$ should be orthogonal to $f_0(h^{-1}g)$ (due to the invariance of $H_2(G, d\mu)$) for any $h \in G$. Then $P(f) = [f * f_0]_l = 0$ and we have shown the orthogonality again. This completes the proof. □

Remark 5.9. The stated left invariance of $H_2(G, d\mu)$ and the representation of P as a right group convolution have a useful tie with differential equations. Really, let X_j, $j \leq m$ be left-invariant vector fields (i. e. left-invariant differential operators) on G. If $X_j f_0 \equiv 0$ then $X_j f = 0$ for any $f \in H$. Thus space $H_2(G, d\mu)$ may be characterized as the space of solutions to the system of equations $X_j f = 0$, $1 \leq j \leq m$.

Another connected formulation: we can think of P as of an integral operator with the kernel $K(h, g) = f_0(g^{-1}h) = \overline{f_0(h^{-1}g)} = \langle T_g f_0, T_h f_0 \rangle$, then kernel $K(h, g)$ is an *analytic* function of h and *anti-analytic* of g.

Example 5.10. An important class of applications may be treated as follows. Let we have a space of functions defined on a domain $\Omega \in \mathbb{R}^n$. Let we have also a transitive Lie group G of automorphisms of Ω. Then we can construct a unitary representation T of G on $L_2(\Omega)$ by the formula:

$$T_g : f(x) \mapsto f(g(x))\, J_g^{1/2}(x),\ f(x) \in L_2(\Omega),\ g \in G, \tag{5.11}$$

where $J_g(x)$ is the Jacobian of the transformation defined by g at the point x.

If we fix some point $x_0 \in \Omega$ then we can identify the homogeneous space G/G_{x_0} with Ω (see notation at the beginning of Subsection 2.1). Then left-invariant vector fields on G may be considered as differential operators on Ω and convolution operators on G as integral operators on Ω. This is a way giving *integral representations for analytic functions*.

5.2. **Classical Results.** We would like to show, how abstract Theorem 5.8 and Example 5.10 are connected with classical results on the Bargmann, Bergman and Szegö projectors at the Segal-Bargmann (Fock), Bergman and Hardy spaces respectively. We will start from a trivial example.

Corollary 5.11. *Let $\{\phi_j\}$, $-\infty < j < \infty$ be an orthonormalized basis of a Hilbert space H. Then*

$$B = \sum_{j=-\infty}^{\infty} |\phi_j\rangle \langle \phi_j| \tag{5.12}$$

is a reproducing operator, namely, $Bf = f$ for any $f \in H$.

Proof. We will construct a unitary representation of group $\mathbb{Z}$ on H by its action on the basis:

$$T_k \phi_j = \phi_{j+k},\ k \in \mathbb{Z}.$$

If we equip $\mathbb{Z}$ with the invariant discreet measure $d\mu(k) = 1$ and select the vacuum vector $f_0 = \phi_0$, then all coherent states are exactly the basis vectors: $f_k = T_k\phi_0 = \phi_k$. Equation (5.3) turns to be exactly the Plancherel formula

$$\langle f_1, f_2 \rangle = \sum_{j=-\infty}^{\infty} \langle f_1, T_j f_0 \rangle\, \overline{\langle f_2, T_j f_0 \rangle} = \sum_{j=-\infty}^{\infty} \langle f_1, \phi_j \rangle\, \overline{\langle f_2, \phi_j \rangle}$$

and we have obtained the usual isomorphism of Hilbert spaces $H \cong \ell_2(\mathbb{Z})$ by the formula $f(k) = \langle f, \phi_k \rangle$. Our construction gives

$$\begin{aligned} f(k) &= \sum_{j=-\infty}^{\infty} \overline{\langle f_0, T_j f_0 \rangle} \, \langle f, T_{j+k} f_0 \rangle \\ &= \sum_{j=-\infty}^{\infty} \delta_{0j} \, \langle f, \phi_{j+k} \rangle \\ &= \langle f, \phi_k \rangle . \end{aligned}$$

Thus operator B is really identical on H. Note, that similar construction may be given for a case of not orthonormalized frame. □

In spite of simplicity of this construction, it was the (almost) unique tool to establish of various projectors (see [46, 3.1.4]). Following less trivial Corollaries bring us back to Section 4.

Corollary 5.12. [2] *The Bargmann projector on the Segal–Bargmann space has the kernel*

$$K(z, w) = e^{\bar{w}(z-w)}. \tag{5.13}$$

Proof. Let us define a unitary representation of the Heisenberg group $\mathbb{H}^n$ on $\mathbb{R}^n$ by the formula [52, § 1.1]:

$$g = (t, q, p) : f(x) \to T_{(t,q,p)} f(x) = e^{i(2t - \sqrt{2} qx + qp)} f(x - \sqrt{2} p).$$

As "vacuum vector" we will select the original *vacuum vector* $f_0(x) = e^{-x^2/2}$. Then embedding $L_2(\mathbb{R}^n) \to L_2(\mathbb{H}^n)$ is given by the formula

$$\begin{aligned} \widetilde{f}(g) &= \langle f, T_g f_0 \rangle \\ &= \pi^{-n/4} \int_{\mathbb{R}^n} f(x) \, e^{-i(2t - \sqrt{2} qx + qp)} \, e^{-(x - \sqrt{2} p)^2/2} \, dx \\ &= e^{-2it - (p^2 + q^2)/2} \pi^{-n/4} \int_{\mathbb{R}^n} f(x) \, e^{-((p+iq)^2 + x^2)/2 + \sqrt{2}(p+iq)x} \, dx \\ &= e^{-2it - z\bar{z}/2} \pi^{-n/4} \int_{\mathbb{R}^n} f(x) \, e^{-(z^2 + x^2)/2 + \sqrt{2} zx} \, dx, \end{aligned} \tag{5.14}$$

where $z = p + iq$, $g = (t, p, q) = (t, z)$. Then $\widetilde{f}(g)$ belong to $L_2(\mathbb{H}^n, dg)$. It is easy to see, that for every fixed t_0 function $\breve{f}(z) = e^{z\bar{z}/2} \widetilde{f}(t_0, z)$ belongs to the Segal-Bargmann space, say, is analytic by z and square-integrable with respect the Gaussian measure $\pi^{-n} e^{-z\bar{z}}$. The integral in (5.14) is the well known Bargmann transform [2]. Then the projector $L_2(\mathbb{H}^n, dg) \to L_2(\mathbb{R}^n)$ is a convolution on the $\mathbb{H}^n$

with the kernel

$$\begin{aligned} P(t,q,p) &= \langle f_0, T_g f_0\rangle \\ &= \pi^{-n/4}\int_{\mathbb{R}^n} e^{-x^2/2}\, e^{-i(2t-\sqrt{2}qx+qp)}\, e^{-(x-\sqrt{2}p)^2/2}\,dx \\ &= \pi^{-n/4}\int_{\mathbb{R}^n} e^{-x^2/2-i2t+\sqrt{2}iqx-iqp-x^2/2+\sqrt{2}px-p^2}\,dx \\ &= e^{-i2t-(p^2+q^2)/2}\pi^{-n/4}\int_{\mathbb{R}^n} e^{-(x-(p+iq)/\sqrt{2})^2}\,dx \\ &= \pi^{n/4}e^{-i2t-z\bar{z}/2}. \end{aligned} \tag{5.15}$$

It was shown during the proof of Proposition 4.5 that a convolution with kernel (5.15) defines the usual value of the Bargmann projector with kernel (5.13). □

Corollary 5.13. [46, 3.1.2] *The orthogonal Bergman projector on the space of analytic functions on unit ball $\mathbb{B}\in\mathbb{C}^n$ has the kernel*

$$K(\zeta,\upsilon) = (1-\langle\zeta,\upsilon\rangle)^{-n-1},$$

where $\langle\zeta,\upsilon\rangle=\sum_1^n \zeta_j\bar{\upsilon}_j$ is the scalar product at $\mathbb{C}^n$.

Proof. We will only rewrite material of Chapters 2 and 3 from [46] using our vocabulary. The group of biholomorphic automorphisms $\mathrm{Aut}\,(\mathbb{B})$ of the unit ball $\mathbb{B}$ acts on $\mathbb{B}$ transitively. For any $\phi\in\mathrm{Aut}\,(\mathbb{B})$ there is a unitary operator associated by (5.11) and defined by the formula [46, 2.2.6(i)]:

$$[T_\phi f](\zeta) = f(\phi(\zeta))\left(\frac{\sqrt{1-|\alpha|^2}}{1-\langle\zeta,\alpha\rangle}\right)^{n+1}, \tag{5.16}$$

where $\zeta\in\mathbb{B}$, $\alpha=\phi^{-1}(0)$, $f(\zeta)\in L_2(\mathbb{B})$. Operator T_ϕ from (5.16) obviously preserve the space $H_2(G)$ of square-integrable holomorphic functions on $\mathbb{B}$. The homogeneous space $\mathrm{Aut}\,(\mathbb{B})/G_0$ may be identified with $\mathbb{B}$ [46, 2.2.5]. To distinguish points of these two sets we will denote points of $B=\mathrm{Aut}\,(\mathbb{B})/G_0\cong\mathbb{B}$ by Roman letters (like a,u,z) and points of $\mathbb{B}$ itself by Greek letters (α,υ,ζ correspondingly). We also always assume, that $a=\alpha, u=\upsilon, z=\zeta$ under the mentioned identification.

We select the function $f_0(\zeta)\equiv 1$ as the vacuum vector. The mean value formula [46, 3.1.1(2)] gives us:

$$\begin{aligned} \widetilde{f}(a) &= \langle f(\zeta), T_\phi f_0\rangle \\ &= \left\langle T_{\phi^{-1}} f(\zeta), f_0\right\rangle \\ &= \int_{\mathbb{B}} f(\phi(\zeta))\left(\frac{\sqrt{1-|\alpha|^2}}{1-\langle\zeta,\alpha\rangle}\right)^{n+1} d\nu(\zeta) \\ &= f(\phi(0))\left(\frac{\sqrt{1-|\alpha|^2}}{1-\langle 0,\alpha\rangle}\right)^{n+1} \\ &= f(a)(\sqrt{1-|\alpha|^2})^{n+1}, \end{aligned} \tag{5.17}$$

where $a=\alpha=\phi(0),\ \phi\in B$ and $\widetilde{f}(a)\in L_2(B)$. Particulary

$$\begin{aligned}\widetilde{f}_0(a) &= (\sqrt{1-|\alpha|^2})^{n+1},\\ \widetilde{f}_0(zu) &= \left(\frac{\sqrt{1-|\zeta|^2}\sqrt{1-|\upsilon|^2}}{1-\langle \upsilon,\zeta\rangle}\right)^{n+1}, \end{aligned} \tag{5.18}$$

An invariant measure on B is given [46, 2.2.6(2)] by the expression:

$$d\mu(z)=\frac{d\nu(\zeta)}{(1-|\zeta|^2)^{n+1}}, \tag{5.19}$$

where $d\nu(\zeta)$ is the usual Lebesgue measure on $B\cong\mathbb{B}$. We will substitute expressions from (5.17), (5.18) and (5.19) to the reproducing formula (5.9):

$$\begin{aligned}\widetilde{f}(u) &= f(\upsilon)(\sqrt{1-|\upsilon|^2})^{n+1} \qquad (5.20)\\ &= \int_B \widetilde{f}(z)\widetilde{f}_0(z^{-1}u)\,d\mu(z)\\ &\overset{(*)}{=} \int_B \widetilde{f}(z)\widetilde{f}_0(zu)\,d\mu(z)\\ &= \int_{\mathbb{B}} f(\zeta)(\sqrt{1-|\zeta|^2})^{n+1}\left(\frac{\sqrt{1-|\zeta|^2}\sqrt{1-|\upsilon|^2}}{1-\langle \upsilon,\zeta\rangle}\right)^{n+1}\frac{d\nu(\zeta)}{(1-|\zeta|^2)^{n+1}}\\ &= \int_{\mathbb{B}} f(\zeta)\left(\frac{\sqrt{1-|\upsilon|^2}}{1-\langle \upsilon,\zeta\rangle}\right)^{n+1} d\nu(\zeta)\\ &= (\sqrt{1-|\upsilon|^2})^{n+1}\int_{\mathbb{B}}\frac{f(\zeta)}{(1-\langle \upsilon,\zeta\rangle)^{n+1}}\,d\nu(\zeta). \qquad (5.21)\end{aligned}$$

Here transformation $(*)$ is possible because every element of B is an involution [46, 2.2.2(v)]. It immediately follows from the comparison of (5.20) and (5.21) that:

$$f(\upsilon)=\int_{\mathbb{B}}\frac{f(\zeta)}{(1-\langle \upsilon,\zeta\rangle)^{n+1}}\,d\nu(\zeta).$$

The last formula is the integral representation with the Bergman kernel for holomorphic functions on unit ball in $\mathbb{C}^n$. □

Corollary 5.14. [23] *The orthogonal projector Szegö on the boundary $b\mathbb{U}^n$ of the upper half-space in $\mathbb{C}^{n+1}$ has the kernel*

$$S(z,w)=(\frac{i}{2}(\bar{w}_{n+1}-z_{n+1})-\sum_{j=1}^{n} z_j\bar{w}_j)^{-n-1}.$$

Proof. It is well known [23, 24, 49] and was described at Example 4.2, that there is a unitary representation of the Heisenberg group $\mathbb{H}^n$ as the simply transitive acting group of shift on $b\mathbb{U}^n$ (see [49, Chap. XII, § 1.4]):

$$(\zeta,t):(z',z_{n+1})\mapsto(z'+\zeta,z_{n+1}+t+2i\,\langle z',\zeta\rangle+i\,|\zeta|^2), \tag{5.22}$$

where $(\zeta,t)\in\mathbb{H}^n, z=(z',z_{n+1})\in\mathbb{C}^{n+1},\zeta,z'\in\mathbb{C}^n,t\in\mathbb{R}$. We again apply the general scheme from Example 5.10. This gives an identification of $\mathbb{H}^n$ and

$b\mathbb{U}^n$ and $\mathbb{H}^n$ act on $b\mathbb{U}^n \cong \mathbb{H}^n$ by left group shifts. Left invariant vector fields are exactly the *tangential Cauchy-Riemann equations* for holomorphic functions on $\mathbb{U}^n$. Shifts (5.22) commute with the tangential Cauchy-Riemann equations and thus preserve the Hardy space $H_2(b\mathbb{U}^n)$ of boundary values of functions holomorphic on $\mathbb{U}^n$.

As vacuum vector we select the function $f_0(z) = (iz_{n+1})^{-n-1} \in H_2(b\mathbb{U}^n)$. Then the Szegö projector $P : \ell_2(b\mathbb{U}^n) \to H_2(b\mathbb{U}^n)$ is the right convolution on $\mathbb{H}^n \cong b\mathbb{U}^n$ with $f_0(z)$ and thus should have the kernel (see the group low formula (2.6) for $\mathbb{H}^n$)

$$S(z, w) = (\frac{i}{2}(\bar{w}_{n+1} - z_{n+1}) - \sum_{j=1}^{n} z_j \bar{w}_j)^{-n-1}.$$

Reader may ask, *why have we selected such a vacuum vector?* The answer is: for a *simplicity* reason. Indeed, the *Cayley transform* ([49, Chap. XII, § 1.2] and [46, § 2.3])

$$C(z) = i\frac{e_{n+1} + z}{1 - z_{n+1}},\ e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{C}^{n+1} \tag{5.23}$$

establishes a biholomorphic map from unit ball $\mathbb{B} \in \mathbb{C}^{n+1}$ to domain $\mathbb{U}^n$. We can construct an isometrical isomorphism of the Hilbert spaces $H_2(\mathbb{S}^{2n+1})$ and $H_2(\mathbb{U}^n)$ based on (5.23)

$$f(z) \mapsto [Cf](z) = f(C(z))\frac{-2i^{n+1} z_{n+1}}{(1 - z_{n+1})^{n+2}},\ f \in H_2(\mathbb{U}^n),\ [Cf] \in H_2(\mathbb{S}^{2n+1}). \tag{5.24}$$

Then the vacuum vector $f_0 = (iz_{n+1})^{-n-1}$ is the image of function $\widetilde{f}_0(w) = (-2i/(w - i))^{n+2} \in H_2(\mathbb{S}^{2n+1})$ under transformation (5.24). It seems to be one from the simplest functions from $H_2(\mathbb{S}^{2n+1})$ with singularities on $\mathbb{S}^{2n+1}$. □

5.3. **Connections with Relative Convolutions.** Now we return to relative convolutions and will show their connections with coherent states. For any operator $A : L_2(G, d\mu) \to L_2(G, d\mu)$ we can construct the *Toeplitz-like* operator $P_A = PA : H_2(G, d\mu) \to H_2(G, d\mu)$. Of course, using the isomorphism $H \cong H_2(G, d\mu)$ we can think about P_A as an operator $P_A : H \to H$. Particulary, operators of group convolution on G will induce relative convolutions on H (see Lemma 3.7). Thus we again have a direct way for applications of harmonic analysis in every problem concerning coherent states. Although, coherent states are very useful in physics we will stop here[10] and will only develop this theme in the next Example connected with wavelets.

6. Applications to Physics and Signal Theory

We are going to consider some Examples connected with physics, but our division between mathematics and physics is so fragile as in the real life.

Example 6.1. Let us consider the "$ax + b$ group" [52, § 7.1] of affine transformations of the real line. We will denote this group by A and its Lie algebra by $\mathfrak{a}$. We will consider their operation on the real line $S = \mathbb{R}$. $\mathfrak{a}$ is spanned by two vector

[10] However, let us remind again that Example 4.4 forms an interesting application to physics.

fields $X_s = \frac{1}{i}\frac{\partial}{\partial y}$ (which generate shifts) and $X_d = y\frac{1}{i}\frac{\partial}{\partial y}$ (which generate dilations). Their commutators are $[X_s, X_d] = X_s$. Then transformation (5.4) takes the form

$$\begin{aligned}\widetilde{f}(x_1, x_2) &= \frac{1}{2\pi}\int_{\mathbb{R}} \bar{f}(y)\, e^{i(x_1 X_s + x_2 X_d)} f_0(y)\, dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \bar{f}(y)\, e^{-x_2/2} f_0(e^{-x^2} y - x_1)\, dy. \end{aligned} \tag{6.1}$$

The last line is easily recognized as the *wavelet transform* [26].

Similar expression in the spirit of Definition 5.2 for the *Gabor transform* [26] may be obtained if we replace A by the Heisenberg group $\mathbb{H}^3$. Then, as was shown early, different signal alterations constructed in signal theory are relative convolution on $S = \mathbb{R}$ induced by $\mathbb{H}^3$ or the meta-Heisenberg group from [22]. For example, signal filtration may be presented at the Gabor representation as a multiplication by characteristic functions of the wished time and frequency intervals.

Using relative convolutions we can coherently introduce wavelets-like transform for every semi-direct product [52, § 5.3] of Lie group and Abelian one [10]. In view of applications to the signal theory it seems interesting to start from the Heisenberg group and its dilations. The one-parameter group $D = \{\delta_\tau \mid \tau \in \mathbb{R}\}$ of dilations of the Heisenberg group is given by the formula

$$\delta_\tau(t, z) = (e^{2\tau} t, e^\tau z),\ (t, z) \in \mathbb{H}^n \cong \mathbb{R} \times \mathbb{R}^{2n} \tag{6.2}$$

and has the one-dimensional Lie algebra spanned on the vector field

$$X_h = \frac{1}{i}(2t\frac{\partial}{\partial t} + z\frac{\partial}{\partial z}). \tag{6.3}$$

If we introduce now the relative convolutions for Lie algebra generated by vector fields X_j^l from (2.10) and X_h from (6.3) then we obtain the *Heisenberg Gabor-like transform*, which should be useful to analyze of the radar ambiguity function [21, § 1.4].

Example 6.2. We are going to describe *group quantization* from paper [37]. The usual “quantization” means some (more or less complete) set of rules for the construction of a quantum algebra from the classical description of a physical system. The group quantization is based on the Hamilton description and consists of the following steps.

(1) Let $\Omega = \{x_j\}, 1 \leq j \leq N$ be a set of physical quantities defining state of a classical system. Observables are real valued functions on the states.

The most known and important case is the set $\{x_j = q_j, x_{j+n} = p_j\}$, $1 \leq j \leq n, N = 2n$ of coordinates and impulses of classical particle with n degrees of freedom. Observables are real valued functions on $\mathbb{R}^{2n}$. This example will be our main illustration during the present consideration.

(2) We will complete the set Ω till $\bar{\Omega}$ by additional quantities $x_j, N < j \leq \bar{N}$, such that $\bar{\Omega}$ will form the smallest algebra containing Ω and closed under the Poisson bracket:

$$\{x_i, x_j\} \in \bar{\Omega}, \quad \text{for all } x_i, x_j \in \bar{\Omega}.$$

In the case of a particle we should add the function $x_{2n+1} = 1$, which is equal to unit identically and one obtains the famous relations ($\bar{N} = 2n+1$)

$$\{x_j, x_{j+n}\} = -\{x_{j+n}, x_j\} = x_{2n+1} \tag{6.4}$$

and all other Poisson brackets are equal to zero.

(3) We form a $\bar{N}$-dimension Lie algebra $\mathfrak{p}$ with a frame $\{\widehat{x}_j\}$, $1 \le j \le \bar{N}$ with the formal mapping $\widehat{}: x_j \mapsto \widehat{x}_j$. Commutators of frame vectors of $\mathfrak{p}$ are formally defined throughout the formula

$$[\widehat{x}_i, \widehat{x}_j] = \widehat{\{x_i, x_j\}} \tag{6.5}$$

and we extend the commutator on whole algebra by the linearity.

For a particle this step give us the Lie algebra $\mathfrak{h}_n$ of the Heisenberg group (compare (2.12) and (6.4)).

(4) We introduce an algebra $\mathfrak{P}$ of relative convolutions (2.4) induced by $\mathfrak{p}$. These operators are *observables* in the group quantization and by analogy to classic case they may be treated as functions of $\widehat{x}_j$ (see Remark 2.4). A set S which algebra $\mathfrak{p}$ acts on and type of kernels are depending on physically determining constraints. The family of all one-dimensional representations of $\mathfrak{P}$ is called *classical* mechanics and different noncommutative representations correspond to *quantum* descriptions with the different *Planck constants*.

For particle we have the following opportunities:

(a) $S = \mathbb{R}^n$, $\widehat{x}_j = X_j = M_{q_j}$, $\widehat{x}_{j+n} = \hbar \frac{1}{i} \frac{\partial}{\partial q_j}$, relative convolutions are PDO from Example 4.1 and we have obtained the *Dirac–Heisenberg–Schrödinger–Weyl quantization* by PDO.

(b) $S = \mathbb{R}^{2n}$, $\widehat{x}_j = X_j = M_{q_j}$, $\widehat{x}_{j+n} = M_{p_j}$, relative convolutions are operators of multiplication by functions (or just functions) from Example 2.6 and we have obtained the usual classical description, which we have started from.

(c) $S = \mathbb{H}^n$, $\widehat{x}_j = X_j^{l(r)}$, $0 \le j \le 2n+1$ and relative convolutions form the group convolution algebra on $\mathbb{H}^n$. This description (so-called *plain mechanics*) contain both the *quantum* and *classical* ones with the natural realization of the *correspondence principle* (see [37] for details).

Group quantization is straightforward enough and obviously preserves the symmetry group of the classical system under investigation. Moreover, there is also other advantages of the proposed quantization, which distinguish it from the already known ones.

- In contrary to the *operator quantization* of Berezin [5] and the *geometrical quantization* of Kirillov–Souriau–Konstant [55] we should not introduce *a priori* any Planck constants. Moreover, during the posterior analysis of relative convolution algebra representations a parameter corresponding to the Planck constant will appear naturally. By the way, a *set of Planck constants* should not necessary belong to $[0, +\infty[$ and may form more complicated topological spaces.
- The problem of ordering of noncommutative quantities $\widehat{q}_j$ and $\widehat{p}_j$ does not occur under group quantization. Correspondence

$$\text{function } k(x) \;\rightarrow\; \text{convolution with the kernel } k(x)$$

is direct enough even for noncommutative groups. Meanwhile, in other quantization the "painful question of ordering" [7] has generated many different answers: the $\widehat{qp}$-quantization, the $\widehat{pq}$-quantization, the Weyl-symmetrical, the Wick and the anti-Wick (Berezin) quantization.

Remark 6.3. The presented group quantization has deep roots in the quantization procedure of Dirac [15]. The main differences are

- We recognize the Heisenberg commutation relations (2.12) only as a particular case among other possibilities. However, due to Theorem 3.8 they play the fundamental role.
- We do not look only for irreducible representations of commutation relations.

7. Conclusion

The paper tried to illustrate *how the systematical usage of harmonic analysis in various applications may be useful for both: the analysis and applications*. It seems, that relative convolutions form an appropriate tool for this purpose.

Given Examples from different fields of mathematics and physics made reasonable studying of relative convolutions. Moreover, we have repeatedly met nilpotent Lie groups (and particularly the Heisenberg) group within important applications, so our primary interest in such groups should be excused.

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